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Extended superconformal symmetry and Calogero–Marchioro model

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Abstract
We show that the two-dimensional Calogero–Marchioro model (CMM) without the harmonic confinement can naturally be embedded into an extended $SU(1,1|2)$ superconformal Hamiltonian. We study the quantum evolution of the superconformal Hamiltonian in terms of suitable compact operators of the $\mathcal{N}=2$ extended de Sitter superalgebra with central charge and discuss the pattern of supersymmetry breaking. We also study the arbitrary D-dimensional CMM having dynamical $OSp(2|2)$ supersymmetry and point out the relevance of this model in the context of the low energy effective action of the dimensionally reduced Yang–Mills theory.

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1. Introduction

The Calogero–Moser–Sutherland (CMS) system is a class of exactly solvable models in one dimension [1–5]. These models have been studied extensively from the time of their inception more than 30 years ago and are well understood. There are many higher-dimensional generalizations of these models [6–13]. Unfortunately, not a single one of these models are known to be exactly solvable or integrable. Among all these systems, the two-dimensional Calogero–Marchioro model (CMM) deserves special attention for several reasons. First of all, for a certain value of the coupling constant, different n-point correlation functions can be calculated analytically by mapping this model to a complex random matrix theory [7, 13, 14]. This model also has been studied extensively [13, 15, 16] in connection with several condensed matter systems such as the quantum Hall effect, quantum dots, two-dimensional Bose systems etc, revealing many interesting features.

The purpose of this paper is to unveil one more new feature of this model. We first study the D-dimensional N-particle super-CMM with ND bosonic and ND fermionic degrees of freedom. We show how infinitely many exact eigenstates can be constructed, both in supersymmetry-breaking and supersymmetry-preserving phases, using the dynamical $OSp(2|2)$ symmetry of the model. We then show that, within the specific formalism, only the two-dimensional CMM without the harmonic confinement can naturally be embedded into an extended superconformal Hamiltonian. In other words, we construct an extended $\mathcal{N} = 2$ superconformal version of the two-dimensional CMM. This construction is valid for arbitrary values of the coupling constant and also for arbitrary N number of particles. We study the quantum evolution in terms of suitable compact operators of the extended $\mathcal{N} = 2$ de Sitter superalgebra. Though we are able to find an infinite number of exact eigenstates of these super-operators, the set is not complete and we are unable to find the complete spectrum. We also discuss the supersymmetry-breaking pattern of the extended $\mathcal{N} = 2$ de Sitter supersymmetry with central charge and show how the half or the complete breakdown of supersymmetry occurs. Finally, we point out the relevance of our findings in the context of super-Yang–Mills (YM) theory.

We organize the paper in the following way. We first introduce the conformal CMM model in arbitrary dimensions in the next section. An infinite number of excited eigenstates corresponding to the radial excitations are constructed algebraically using the underlying $SU(1, 1)$ symmetry. We construct the superconformal CMM in arbitrary dimensions in section 3. We also obtain infinitely many exact eigenstates using the dynamical $OSp(2|2)$ symmetry of the model. The extended $\mathcal{N} = 2$ superconformal CMM in $D = 2$ is constructed in section 4. The symmetry algebra of the model and the supersymmetry-breaking pattern is discussed. Finally, in section 5, we summarize our findings and discuss the relevance of our results. We point out a possible relation between the D-dimensional CMM considered in this paper and the low-energy effective action of the $D + 1$-dimensional YM theory dimensionally reduced to $0 + 1$ dimension.

2. Conformal CMM

We first consider the three operators h, $\mathcal{D}$ and K given by

$$\begin{aligned} h &= \frac{1}{2}\sum_{i,\mu} p_{i,\mu}^2 + \frac{g}{2}(g + D - 2)\sum_{i\neq j} \vec{r}_{ij}^{\,-2} + \frac{g^2}{2}\sum_{i\neq j\neq k} (\vec{r}_{ij}\cdot\vec{r}_{ik})\vec{r}_{ij}^{\,-2}\vec{r}_{ik}^{\,-2} \\ \mathcal{D} &= -\tfrac{1}{4}\sum_{i,\mu}\{x_{i,\mu}, p_{i,\mu}\} \qquad K = \tfrac{1}{2}\sum_{i,\mu} x_{i,\mu}^2 \qquad p_{i,\mu} = -\mathrm{i}\frac{\partial}{\partial x_{i,\mu}} \qquad \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \end{aligned} \tag{1}$$

where $\vec{r}_i$ is the D-dimensional position vector of the ith particle with $x_{i,\mu}$ the components and g the coupling constant. We fix the convention that the Roman indices run from 1 to N, while the Greek indices run from 1 to D. These three operators admit the O(2, 1) algebra,

$$[h, \mathcal{D}] = \mathrm{i}h \qquad [h, K] = 2\mathrm{i}\mathcal{D} \qquad [K, \mathcal{D}] = -\mathrm{i}K. \tag{2}$$

For the general conformal Hamiltonian, the many-body interaction of h (the last two terms) should be replaced by a degree -2 homogeneous function of the coordinates. The Hamiltonian h describes the CMM without the harmonic confinement. However, the ground state of h with the ground-state energy $E = 0$ is not even plane-wave normalizable. Following the prescription suggested by de Alfaro *et al* [17] for such a quantum mechanical model with conformal symmetry, the quantum evolution can be described by an appropriate compact operator. This compact operator can be constructed from the linear combination of the Hamiltonian h, the dilatation generator $\mathcal{D}$ and the conformal generator K. Following [17], we choose this compact operator H as $H = h + K$. The introduction of K breaks the scale invariance. The operator H is the D-dimensional CMM.

In one dimension, H is exactly solvable and known as the rational CMS Hamiltonian. In $D \geqslant 2$, though infinitely many exact eigenstates of this Hamiltonian can be found, the complete eigenspectrum is still not known. The ground-state wavefunction is determined as [6, 7]

$$\psi_0 = \prod_{i<j} |\vec{r}_i - \vec{r}_j|^g \mathrm{e}^{-\frac{1}{2}\sum_i \vec{r}_i^2} \tag{3}$$

with the ground-state energy $E_0 = \frac{ND}{2} + gN(N-1)/2$. Using the underlying $SU(1,1)$ symmetry,

$$B_2^{\pm} = -\tfrac{1}{2}(h - K \mp 2\mathrm{i}\mathcal{D}) \qquad [H, B_2^{\pm}] = \pm 2B_2^{\pm} \qquad [B_2^-, B_2^+] = H \tag{4}$$

one can construct infinitely many exact eigenstates of this Hamiltonian. In particular,

$$\psi_n = (B_2^+)^n \psi_0 \tag{5}$$

are exact eigenstates of H with $E_n = E_0 + 2n$. For $D = 3$, these exact eigenstates corresponding to the radial excitations were first obtained in [6] by directly solving the Schrödinger equation. Following the same method, these eigenstates were constructed for arbitrary D in [7]. However, we provide here an algebraic construction of these radial excitations in equation (5), using the underlying $SU(1,1)$ symmetry. Unfortunately, the complete spectrum of H is still not known. The incompleteness of the spectrum can be understood in the following way. In the limit $g \to 0$, the Hamiltonian H reduces to that of a system of N free harmonic oscillators in D dimensions. Thus, in this limit, the complete spectrum of a system of N free oscillators in D dimensions should be reproduced. This is not the case, as can be seen from the expressions ψ_n and E_n given above.

3. $\mathcal{N} = 1$ superconformal CMM: $OSp(2|2)$

We now construct the supersymmetric version of h and H. The supercharge q and its conjugate $q^\dagger$ are defined as

$$q = \sum_{i,\mu} \psi_{i,\mu}^\dagger a_{i,\mu} \qquad q^\dagger = \sum_{i,\mu} \psi_{i,\mu} a_{i,\mu}^\dagger \tag{6}$$

where the ND fermionic variables $\psi_{i,\mu}$ satisfy the Clifford algebra

$$\{\psi_{i,\mu}, \psi_{j,\nu}\} = 0 = \{\psi_{i,\mu}^\dagger, \psi_{j,\nu}^\dagger\} \qquad \{\psi_{i,\mu}, \psi_{j,\nu}^\dagger\} = \delta_{ij}\delta_{\mu,\nu}. \tag{7}$$

The $a_i(a_i^\dagger)$ operators are analogous to bosonic annihilation (creation) operators. They are defined in terms of the momentum operators $p_{i,\mu}$ and the superpotential $W(x_{1,1}, x_{1,2}, \ldots, x_{1,D}, x_{2,1}, \ldots, x_{N,D-1}, x_{N,D})$ as

$$a_{i,\mu} = p_{i,\mu} - \mathrm{i}W_{i\mu} \qquad a_{i,\mu}^\dagger = p_{i,\mu} + \mathrm{i}W_{i,\mu} \qquad W_{i,\mu} = \frac{\partial W}{\partial x_{i,\mu}}. \tag{8}$$

For the general superconformal quantum mechanics, the superpotential should have the following form:

$$W = -\ln G \qquad \sum_{i,\mu} x_{i,\mu} \frac{\partial G}{\partial x_{i,\mu}} = dG \tag{9}$$

where d is any arbitrary constant. We choose the superpotential W as

$$G = \prod_{i<j} |\vec{r}_{ij}|^g \tag{10}$$

which results in the following Hamiltonian:

$$\begin{aligned} h_{\mathrm{s}} &= \tfrac{1}{2}\{q, q^{\dagger}\} \\ &= h + g \sum_{i\neq j;\mu} (2(x_{i,\mu} - x_{j,\mu})^2 \vec{r}_{ij}^{\,-2} - 1)\vec{r}_{ij}^{\,-2}(\psi_{i,\mu}^{\dagger}\psi_{i,\mu} - \psi_{i,\mu}^{\dagger}\psi_{j,\mu}) \\ &\quad + 2g \sum_{i\neq j;\mu\neq\nu} (x_{i,\mu} - x_{j,\mu})(x_{i,\nu} - x_{j,\nu})\vec{r}_{ij}^{\,-4}(\psi_{i,\mu}^{\dagger}\psi_{i,\nu} - \psi_{i,\mu}^{\dagger}\psi_{j,\nu}). \end{aligned} \tag{11}$$

The super-Hamiltonian h_{s} is the supersymmetric generalization of h. This can be checked by projecting h_{s} in the zero-fermion sector ($\psi_{i,\mu}|0\rangle = 0$) of the 2^{DN}-dimensional fermionic Fock space.

The super-Hamiltonian h_{s} does not have a normalizable ground state. Following the standard procedure in the literature [17–19], the quantum evolution can be described by the operators R or H_{s} defined as

$$\begin{aligned} &H_{\mathrm{s}} = R + B - c \qquad R = h_{\mathrm{s}} + K \\ &B = \tfrac{1}{2}\sum_{i,\mu}[\psi_{i,\mu}^{\dagger}, \psi_{i,\mu}] \qquad c = \frac{g}{2}N(N-1). \end{aligned} \tag{12}$$

The new operator H_{s} is the supersymmetric generalization of the D-dimensional CMM H. The complete eigenspectrum of this operator is known [20,21] for $D = 1$, both in supersymmetry-preserving ($g > 0$) as well as supersymmetry-breaking ($g < 0$) phases. No attempt has been made so far to study H_{s} with its full generality for $D \geqslant 2$. We find that the ground state of H_{s} in the supersymmetric phase ($g > 0$) is determined as, $\psi_{\mathrm{s}}^0 = \psi_0|0\rangle$. A comment is in order at this point. The ground-state wavefunction ψ_{s}^0 is normalizable for $g > -\frac{1}{2}$. However, a stronger criterion that each momentum operator $p_{i,\mu}$ is self-adjoint for the wavefunctions of the form ψ_{s}^0 requires $g > 0$. The supersymmetry is preserved for $g > 0$, while it is broken for $g < 0$ [18,21]. Let us now define the following operators:

$$\begin{aligned} &Q_1 = q - \mathrm{i}S \qquad Q_2 = q^{\dagger} - \mathrm{i}S^{\dagger} \qquad S = \sum_{i,\mu} \psi_{i,\mu}{}^{\dagger} x_{i,\mu} \\ &Q_1^{\dagger} = q^{\dagger} + \mathrm{i}S^{\dagger} \qquad Q_2^{\dagger} = q + \mathrm{i}S \qquad S^{\dagger} = \sum_{i,\mu} \psi_{i,\mu} x_{i,\mu}. \end{aligned} \tag{13}$$

Note that the super-Hamiltonian $H_{\mathrm{s}} = \frac{1}{2}\{Q_1, Q_1^{\dagger}\}$. One can define bosonic and fermionic creation operators [18,21]

$$\mathcal{B}_2^{\dagger} = -\tfrac{1}{4}\{Q_1^{\dagger}, Q_2^{\dagger}\} \qquad \mathcal{F}_2^{\dagger} = Q_2^{\dagger}. \tag{14}$$

It can be checked easily that

$$[H_{\mathrm{s}}, \mathcal{B}_2^{\dagger}] = 2\mathcal{B}_2^{\dagger} \qquad [H_{\mathrm{s}}, \mathcal{F}_2^{\dagger}] = 2\mathcal{F}_2^{\dagger}. \tag{15}$$

We construct a set of exact eigenstates with the help of these operators. In particular,

$$\psi_{n,\nu} = \mathcal{B}_2^{\dagger^n} \mathcal{F}_2^{\dagger^\nu} \psi_{\mathrm{s}}^0 \tag{16}$$

are the exact eigenstates of H_{s} with the energy $E_{n,\nu} = 2(n+\nu)$. The bosonic quantum number n can take any non-negative integer values, while the fermionic quantum number $\nu = 0, 1$. The super-Hamiltonian H_{s} reduces to that of N free super-oscillators in D dimensions in the limit $g \to 0$. In the same limit, one would thus expect to obtain the complete eigenspectrum of N free super-oscillators in D dimensions from $\psi_{n,\nu}$ and $E_{n,\nu}$. Unfortunately, $E_{n,\nu}$ and $\psi_{n,\nu}$ describe only a small part of the complete spectrum of the free super-oscillator Hamiltonian. Thus, the set of exact eigenstates (16) is not complete and we are unable to find the complete spectrum.

The supersymmetry-breaking phase of H_s is characterized by $g < 0$. A set of exact eigenstates in this phase can also be constructed by using a duality property of this Hamiltonian. Consider a dual-Hamiltonian $\tilde{H}_s$ constructed in terms of Q_2 and $Q_2^\dagger$ as $\tilde{H}_s = \frac{1}{2}\{Q_2, Q_2^\dagger\}$. This Hamiltonian can also be obtained from H_s by making $g \to -g$ and $\psi_{i,\mu} \leftrightarrow \psi_{i,\mu}^\dagger$ [21]. We determine the ground-state of $\tilde{H}_s$ in its own supersymmetric phase ($g < 0$) as

$$\tilde{\psi}_0 = \prod_{i<j} |\vec{r}_i - \vec{r}_j|^{-g} \mathrm{e}^{-\frac{1}{2}\sum_i \vec{r}_i^{\,2}} |ND\rangle \qquad \psi_{i,\mu}^\dagger |ND\rangle = 0. \tag{17}$$

Note that $\tilde{H}_s$ is related to H_s by the following relation:

$$H_s = \tilde{H}_s + B - 2c. \tag{18}$$

Thus, $\tilde{\psi}_0$ is also an exact eigenstate of H_s with the ground-state energy $E_0 = B - 2c$, which is positive definite for $g < 0$. This is in fact the ground-state wavefunction of H_s in the supersymmetry-breaking phase. A comment is in order at this point. Usually, there are no general methods to find eigenstates in supersymmetry-breaking phase of a model. However, the duality symmetry of H_s plays an important role in understanding the supersymmetry-breaking phase of the model. Firstly, the wavefunction $\tilde{\psi}_0$ is guaranteed to be the ground state of H_s for $g < 0$, because of the relation (18) and the fact that $\tilde{\psi}_0$ is the ground state of the dual-Hamiltonian $\tilde{H}_s$ in its own supersymmetry-preserving phase $g < 0$. Further, an algebraic construction of excited states of H_s for $g < 0$ is possible using the duality symmetry. In particular, a set of excited states can be obtained by applying different powers of the bosonic creation operator $\tilde{\mathcal{B}}_2^\dagger$ and the fermionic creation operator $\tilde{\mathcal{F}}_2^\dagger$ on $\tilde{\psi}_0$, where these operators are obtained from (14) by making $g \to -g$ and $\psi_{i,\mu} \leftrightarrow \psi_{i,\mu}^\dagger$. In particular, the eigenstates and the corresponding eigenvalues are

$$\tilde{\psi}_{n,\nu} = \tilde{\mathcal{B}}_2^{\dagger^n} \tilde{\mathcal{F}}_2^{\dagger^n} \tilde{\psi}_0 \qquad \tilde{E}_{n,\nu} = E_0 + 2(n + \nu). \tag{19}$$

This set of exact eigenstates is, again, not complete.

4. $\mathcal{N} = 2$ superconformal CMM: $SU(1,1|2)$

After the centre-of-mass separation, the super-Hamiltonian h_s for $D = 2$ and $N = 2$ reduces to the model considered in [18]. This model has been shown to have extended $SU(1,1|2)$ superconformal symmetry [18]. We generalize the work of [18] for an arbitrary two-dimensional N particle systems and find the criterion for having $SU(1,1|2)$ superconformal symmetry in the following. The superpotential (9) with the further constraint

$$G = f(z_1, z_2, \ldots, z_N)\, g(z_1^*, z_2^*, \ldots, z_N^*) \qquad z_k = x_{k,1} + \mathrm{i}x_{k,2} \qquad z_k^* = x_{k,1} - \mathrm{i}x_{k,2} \tag{20}$$

always gives rise to $\mathcal{N} = 2$ superconformal Hamiltonian. The homogeneity condition on G implies that the (anti-)holomorphic function $(g)f$ should also be homogeneous. Note that except for the two-dimensional CMM and a nearest-neighbour variant of this model [11], none of the other two-dimensional models [8–10] satisfies the above criterion. Thus, the two-dimensional CMM enjoys a special status over all other models. We specialize to $D = 2$ and CMM in the remainder of the discussion.

Let us define an operator Y and its conjugate $Y^\dagger$ as

$$Y = \frac{1}{2}\sum_i \epsilon_{\mu\nu}\psi_{i,\mu}\psi_{i,\nu} \qquad Y^\dagger = -\frac{1}{2}\sum_i \epsilon_{\mu\nu}\psi_{i,\mu}^\dagger\psi_{i,\nu}^\dagger \tag{21}$$

where $\epsilon_{\mu\nu}$ is the two-dimensional Levi-Civita pseudo-tensor. We follow the convention that the repeated indices of the Levi-Civita pseudo-tensor are always summed over. The operators Y, $Y^\dagger$ and B constitute a $SU(2)$ algebra,

$$[Y, Y^\dagger] = -B \qquad [B, Y] = -2Y \qquad [B, Y^\dagger] = 2Y^\dagger. \tag{22}$$

Further, we have the following commutation relations:

$$[Y^\dagger, \psi_{i,\mu}] = \epsilon_{\mu\nu}\psi^\dagger_{i,\nu} = \bar{\psi}_{i,\mu} \qquad [Y, \psi^\dagger_{i,\mu}] = -\epsilon_{\mu\nu}\psi_{i,\nu} = -\bar{\psi}^\dagger_{i,\mu}. \tag{23}$$

Following [18], it can be shown that the unitary transformation U, which represents a 180° rotation around the second axis in the internal space, performs the following transformation:

$$U^{-1}\psi_{i,\mu}U = \bar{\psi}_{i,\mu} \qquad U^{-1}\psi^\dagger_{i,\mu}U = \bar{\psi}^\dagger_{i,\mu}. \tag{24}$$

The $SU(2)$ generators Y, $Y^\dagger$ and B commute with the Hamiltonian $h_{\rm s}$. The Hamiltonian $h_{\rm s}$ has the internal $SU(2)$ symmetry and is invariant under the unitary transformation U.

The extended $\mathcal{N} = 2$ supersymmetry can be constructed by combining together the $SU(2)$ generators, the operators Q_1, Q_2, S and their conjugates and a set of new operator $\bar{A} = U^{-1}AU$ corresponding to each odd operator A. Define the new supercharges $\bar{q}$ and $\bar{q}^\dagger$ following this prescription as [18]

$$\bar{q} = \sum_{i,\mu} \bar{\psi}^\dagger_{i,\mu}a_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi_{i,\nu}a_{i,\mu} \qquad \bar{q}^\dagger = \sum_{i,\mu}\bar{\psi}_{i,\mu}a^\dagger_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi^\dagger_{i,\nu}a^\dagger_{i,\mu}. \tag{25}$$

These supercharges satisfy the following anticommutation relations [18]:

$$\tfrac{1}{2}\{q, q^\dagger\} = h_{\rm s} \qquad \tfrac{1}{2}\{\bar{q}, \bar{q}^\dagger\} = h_{\rm s}. \tag{26}$$

All other anticommutators among themselves vanish. The super-Hamiltonian will now have a quartet structure. However, as noted earlier, $h_{\rm s}$ does not have a normalizable ground state. The quantum evolution can be described by $R = h_{\rm s} + K$ or $H_{\rm s}$. We now explore the full $SU(1, 1|2)$ symmetry. Define [18]

$$\begin{aligned} &\bar{Q}_1 = \bar{q} - {\rm i}\bar{S} \qquad \bar{Q}_2 = \bar{q}^\dagger - {\rm i}\bar{S}^\dagger \qquad \bar{S} = \sum_{i,\mu}\bar{\psi}^\dagger_{i,\mu}x_{i,\mu} = \sum_i \epsilon_{\mu\nu}\psi_{i,\nu}x_{i,\mu} \\ &\bar{Q}^\dagger_1 = \bar{q}^\dagger + {\rm i}\bar{S}^\dagger \qquad \bar{Q}^\dagger_2 = \bar{q} + {\rm i}\bar{S} \qquad \bar{S}^\dagger = \sum_{i,\mu}\bar{\psi}_{i,\mu}x_{i,\mu} = \sum_i \epsilon_{\mu\nu}\psi^\dagger_{i,\nu}x_{i,\mu}. \end{aligned} \tag{27}$$

The operators Q_1, $\bar{Q}_2$ and their conjugates have the following anticommutator algebra:

$$\begin{aligned} &\tfrac{1}{2}\{Q_1, Q^\dagger_1\} = R + B - c = H_{\rm s} \\ &\tfrac{1}{2}\{\bar{Q}_2, \bar{Q}^\dagger_2\} = R + B + c = H_{\rm s} + 2c \\ &\tfrac{1}{2}\{Q_1, \bar{Q}^\dagger_2\} = -\tfrac{1}{2}\{Q^\dagger_1, \bar{Q}_2\} = -{\rm i}J \end{aligned} \tag{28}$$

where the angular momentum operator is defined as [18]

$$J = \sum_i \epsilon_{\mu\nu}(x_{i,\nu}p_{i,\mu} + {\rm i}\psi^\dagger_{i,\mu}\psi_{i,\nu}). \tag{29}$$

Similarly, the only non-vanishing anticommutators among $\bar{Q}_1$, Q_2 and their conjugates are

$$\begin{aligned} &\tfrac{1}{2}\{Q_2, Q^\dagger_2\} = R - B + c = \tilde{H}_{\rm s} \\ &\tfrac{1}{2}\{\bar{Q}_1, \bar{Q}^\dagger_1\} = R - B - c = \tilde{H}_{\rm s} - 2c \\ &\tfrac{1}{2}\{Q_2, \bar{Q}^\dagger_1\} = -\tfrac{1}{2}\{Q^\dagger_2, \bar{Q}_1\} = -{\rm i}J. \end{aligned} \tag{30}$$

All other non-vanishing anticommutators are given by

$$\begin{aligned} &-\tfrac{1}{2}\{Q_1, \bar{Q}^\dagger_1\} = \tfrac{1}{2}\{\bar{Q}_2, Q^\dagger_2\} = 2Y^\dagger \qquad -\tfrac{1}{2}\{\bar{Q}_1, Q^\dagger_1\} = \tfrac{1}{2}\{Q_2, \bar{Q}^\dagger_2\} = 2Y \\ &\tfrac{1}{4}\{Q_1, Q_2\} = \tfrac{1}{4}\{\bar{Q}_1, \bar{Q}_2\} = -\mathcal{B}_2 \qquad \tfrac{1}{4}\{Q^\dagger_1, Q^\dagger_2\} = \tfrac{1}{4}\{\bar{Q}^\dagger_1, \bar{Q}^\dagger_2\} = -\mathcal{B}^\dagger_2. \end{aligned} \tag{31}$$

The evolution can be described either by $H_{\rm s}$ or $\tilde{H}_{\rm s}$.

The supercharges Q_1 and $\bar{Q}_2$ are the generators of an extended $\mathcal{N} = 2$ de Sitter supersymmetry with the central charge c. It is amusing to note that the central charge c is precisely the energy of the classical minimum equilibrium configurations of the bosonic part of H_s. However, we do not find any topological origin of c, as in the case of field theories admitting soliton solutions in the Bogomol'nyi–Prasad–Sommerfeld limit. As mentioned earlier, ψ_s^0 is the ground state of H_s in the supersymmetric phase. This essentially implies that the supersymmetry associated with the generator $\bar{Q}_2$ has broken. Thus, this is the case corresponding to the spontaneous breakdown of supersymmetry from $\mathcal{N} = 2 \to \mathcal{N} = 1$. For $g < 0$, the supersymmetry spontaneously breaks down completely. The eigenspectrum of H_s in this supersymmetry-breaking phase can be constructed from $\tilde{H}_s$.

The anticommutator algebra (28) is not in diagonal form because of the last equation. The eigenstates of H_s correspond to the angular momentum eigenvalue $j = 0$. Following [18] exactly, let us define

$$\mu = \cos\theta Q_1 + \mathrm{i}\sin\theta \bar{Q}_2 \qquad \nu = \mathrm{i}\sin\theta Q_1 + \cos\theta \bar{Q}_2 \tan(2\theta) = j/c. \tag{32}$$

It can be checked easily that

$$\tfrac{1}{2}\{\mu, \mu^\dagger\} = R + B - \sqrt{c^2 + j^2} \qquad \tfrac{1}{2}\{\nu, \nu^\dagger\} = R + B + \sqrt{c^2 + j^2} \qquad \{\mu, \nu^\dagger\} = 0. \tag{33}$$

The condition that the supersymmetric ground state is annihilated by both μ and $\mu^\dagger$ gives

$$\begin{aligned} \psi_0^s(j) &= \prod_{i<j} (z_i - z_j)^{g^-} (z_i^* - z_j^*)^{g^+} \mathrm{e}^{-\frac{1}{2}\sum_i z_i z_i^*} |0\rangle \\ g^\mp &= \frac{1}{N(N-1)} [(j^2 + c^2)^{\frac{1}{2}} \mp j]. \end{aligned} \tag{34}$$

Note that for $j = 0$, $g^+ = g^- = \frac{g}{2}$ and $\psi_0^s(j = 0)$ reduces to ψ_0^s. The eigenstates in (34) carry an angular momentum,

$$j = \tfrac{1}{2}(g_+ - g_-)N(N - 1). \tag{35}$$

Note that j receives contribution only from the bosonic part of $\psi_0^s(j)$. The rest of the analysis can be carried out in a straightforward way. In particular, one can easily verify that

$$H_s(j) = \tfrac{1}{2}\{\mu, \mu^\dagger\} \qquad [H_s(j), \mathcal{B}_2^\dagger] = 2\mathcal{B}_2^\dagger \qquad [H_s(j), \mathcal{F}_2^\dagger] = 2\mathcal{F}_2^\dagger. \tag{36}$$

Thus, we construct the excited states as

$$\psi_{n,\nu}(j) = \mathcal{B}_2^{\dagger\, n} \mathcal{F}_2^{\dagger\, \nu} \psi_0^s(j) \tag{37}$$

where the bosonic quantum number n can take any non-negative integer values, while the fermionic quantum number $\nu = 0, 1$. Note that all these eigenstates have the same angular momentum.

5. Summary and discussion

We have constructed and studied the D-dimensional superconformal CMM having dynamical $OSp(2|2)$ symmetry. Though we have obtained an infinite number of exact states corresponding to the bosonic and the fermionic excitations, the complete spectrum is still not known. Further, we have shown that the two-dimensional CMM can naturally be embedded into an extended $SU(1, 1|2)$ superconformal Hamiltonian. This construction of extended $\mathcal{N} = 2$ superconformal many-particle Hamiltonian is valid for arbitrary number of particles and also for arbitrary values of the coupling constant. This is the central result of our paper. We have also studied the evolution of this system in terms of operators of the extended $\mathcal{N} = 2$ de Sitter supersymmetry and discussed the supersymmetry-breaking pattern.

It may be worth mentioning here that an attempt to construct one-dimensional CMS Hamiltonian with extended superconformal symmetry has been made recently [22]. It is found that within the specific formalism, the $SU(1, 1|2)$ superconformal CMS model in one dimension can be constructed only for a certain value of the coupling constant. Further, though a general formulation of the multidimensional supersymmetric quantum mechanics with $\mathcal{N} = 2$ was given in [23], no nontrivial many-particle systems of CMS type have yet been shown to result from such formulation. To the best of our knowledge, we are not aware of any other work discussing the $SU(1, 1|2)$ superconformal Hamiltonian of CMM type with its full generality. Within this background, the extended $\mathcal{N} = 2$ superconformal CMM presented in this paper appears to be the first such example in the literature. The space-time dimensionality plays an obvious role in our analysis. However, we would like to stress again that only the CMM and a nearest-neighbour variant of this model [11], among several other interesting many-particle two-dimensional models [8–10], are amenable for such a construction.

The history of studying the supersymmetric quantum mechanical model with higher number of supercharges [24] is long. One of the major reasons for the renewed interest in the (super-)conformal quantum mechanics is its relevance in the study of adS/CFT correspondence and black holes [25]. Though a direct connection between the CMM and the black hole physics cannot be established at this point, we observe a possible relation between the D-dimensional CMM and the low-energy effective action of $D + 1$-dimensional YM theory dimensionally reduced to $0+1$ dimension. This observation is based on the existing results on this subject in the literature [12, 13].

It is known [7, 13, 14] that the Hamiltonian h for $D = 2$ and $g = \frac{1}{2}$ describes the dynamics of a Gaussian ensemble of $N \times N$ normal matrices in the limit $N \to \infty$. The Gaussian action of the normal matrices is given by

$$\mathcal{A}(M, M^\dagger) = \frac{1}{4} \int \mathrm{d}t \,\mathrm{Tr} \left(\frac{\partial M^\dagger}{\partial t} \frac{\partial M}{\partial t} \right) \qquad [M, M^\dagger] = 0. \tag{38}$$

The second equation defines M to be normal matrices. The action $\mathcal{A}$ with M as normal matrices is the low-energy effective action of $2 + 1$-dimensional YM theory dimensionally reduced to $0 + 1$ dimension with the choice of gauge $A_0 = 0$ [12]. A term of the form $[M, M^\dagger]^2$ drops out in the low-energy limit giving rise to the constraint on M to be normal matrices. Thus, for the first time in the literature, we observe the relation between the two-dimensional CMM with $g = \frac{1}{2}$ and the low-energy effective action of $2+1$-dimensional YM theory dimensionally reduced to $0 + 1$ dimension. It is desirable to extend this result for arbitrary value of g, much akin to the one-dimensional CMS system.

It is worth recalling that an attempt to construct higher-dimensional generalizations of the one-dimensional CMS system from many-matrix models has been made in [12]. At the classical level, the resulting Hamiltonian contains only a two-body interaction term of the form $\sum_{i \neq j} \vec{r}_{ij}^{\,-2}$. No trace of a three-body term as in h has been found. However, for $D = 2$, the many-matrix model considered in [12] is identical to $\mathcal{A}$ with M as normal matrix which reduces to CMM with $g = \frac{1}{2}$ in the quantum mechanical treatment [13]. Thus, it is expected that the highly constrained classical models considered in [12] should give rise to the CMM upon quantization for $D = 2$. We also expect that this will provide us a connection between the low-energy effective action of $2+1$-dimensional YM theory dimensionally reduced to $0+1$ dimension and the two-dimensional CMM for arbitrary value of g. Based on this observation, we believe that the D-dimensional super-CMM considered in this paper is in fact related to the low-energy effective action of the $D+1$-dimensional super-YM theory dimensionally reduced to $0 + 1$ dimension. Since the dimensionally reduced super-YM theory appears in many areas of recent research activity like M-theory, D0-branes etc [26], it is of immense interest to put our belief relating CMM and super-YM on a firm footing.

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